Topic 10 -
Reduction of order

In this section we give a specialized technique for finding a second solution to

$$a_2(x)y'' + a_1(x)y' + a_0 y = 0$$

when $a_2(x) \neq 0$ for all x in I and you already know one solution to this equation.

Since we are assuming that $a_2(x) \neq 0$ for all x in I we can divide by it and assume our equation is of the form

$$y'' + a_1(x)y' + a_0(x)y = 0 \qquad (*)$$

Suppose y_1 is a known solution to $(*)$ and further assume $y_1(x) \neq 0$ for all x in I.

Let $y_2(x) = v(x) \cdot y_1(x)$.

We want to find v so y_2 also solves $(*)$.

We know by assumption that

$$y_1'' + a_1(x) y_1' + a_0(x) y_1 = 0$$

Since $y_2 = v \cdot y_1$ we get

$$y_2' = v' y_1 + v y_1'$$

$$y_2'' = v'' y_1 + v' y_1' + v' y_1' + v y_1''$$

$$= v'' y_1 + 2v' y_1' + v y_1''$$

Subbing these into $(*)$ we want to find v such that

$$(v'' y_1 + 2v' y_1' + v y_1'') + a_1(x)(v' y_1 + v y_1') + a_0(x) v y_1 = 0$$

Rearranging we want

$$v'' y_1 + v'(2y_1' + a_1(x) y_1) + v(\underbrace{y_1'' + a_1(x) y_1' + a_0(x) y_1}_{0}) = 0$$

This reduces to needing to find v where

$$v'' y_1 + v'(2y_1' + a_1(x) y_1) = 0$$

This becomes

$$\frac{v''}{v'} = \frac{-2y_1' - a_1(x) y_1}{y_1}$$

which is

$$\frac{v''}{v'} = \frac{-2y_1'}{y_1} - a_1(x)$$

Integrating with respect to x gives

$$\ln(v') = -\ln(y_1^2) - \int a_1(x)\,dx$$

This gives

$$v' = e^{-\ln(y_1^2) - \int a_1(x)\,dx}$$

$$v' = \frac{1}{y_1^2} \cdot e^{-\int a_1(x)\,dx}$$

$$v = \int \frac{e^{-\int a_1(x)\,dx}}{y_1^2}\,dx$$

And $y_2 = v \cdot y_1$.

Note that y_1 and y_2 will be linearly independent because

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} y_1 & vy_1 \\ y_1' & v'y_1 + vy_1' \end{vmatrix}$$

$$= y_1 v' y_1 + y_1 v y_1' - y_1' v y_1$$

$$= y_1^2 v'$$

$$= e^{-\int a_1(x)\,dx} \neq 0$$

<u>Summary:</u> Let $a_1(x)$ be continuous on I.

Let y_1 be a solution to

$$y'' + a_1(x) y' + a_0(x) y = 0$$

on I where $y_1(x) \neq 0$ for all x in I.

Then,

$$y_2 = y_1 \cdot \int \frac{e^{-\int a_1(x)dx}}{y_1^2}\, dx$$

will be another solution that is linearly independent with y_1

Ex: Consider the equation

$$(x^2+1)\,y'' - 2xy' + 2y = 0 \qquad (**)$$

$$\underbrace{(x^2+1)}_{a_2(x) = x^2+1 \neq 0 \text{ for all } x}$$

One can guess that $y_1 = x$ is a solution to $(**)$.

Rewrite $(**)$ as

$$y'' - \underbrace{\frac{2x}{x^2+1}}_{a_1(x) = \frac{-2x}{x^2+1}} y' + \frac{2}{x^2+1}\, y = 0$$

Another solution will be

$$y_2 = y_1 \cdot \int \frac{e^{-\int a_1(x)\,dx}}{y_1^2}\, dx$$

$$= x \cdot \int \frac{\left(e^{-\int \frac{-2x}{x^2+1}\, dx}\right)}{x^2}\, dx$$

$$= x \cdot \int \frac{e^{\ln(x^2+1)}}{x^2}\, dx$$

$$\int \frac{2x}{x^2+1}\, dx = \int \frac{1}{u}\, du = \ln|u|$$

$$u = x^2+1$$

$$du = 2x\, dx$$

$$= \ln|x^2+1|$$

$$= \ln(x^2+1)$$

$$= x \cdot \int \frac{x^2+1}{x^2}\, dx$$

$$= x \cdot \int (1 + x^{-2})\, dx$$

$$= x \left(x + \frac{x^{-1}}{-1} \right)$$

$$= x^2 - 1$$

Thus, $y_1 = x$ and $y_2 = x^2 - 1$ are two linearly independent solutions to $(**)$.

Thus every solution to (**) is of the form

$$y = c_1 y_1 + c_2 y_2 = c_1 x + c_2 (x^2 - 1).$$